Tuesday, September 15, 2015

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Problem 1

Problem. Find the indefinite integral $\int \frac{dx}{\sqrt{9-x^2}}$.

Solution. Let $x = 3u$ and $dx = 3\,du$. Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3\,du}{\sqrt{9-9u^2}} \\
&= \int \frac{3\,du}{3\sqrt{1-u^2}} \\
&= \int \frac{du}{\sqrt{1-u^2}} \\
&= \arcsin u + C \\
&= \arcsin \frac{x}{3} + C.
\end{aligned}$$

Problem 2

Problem. Find the indefinite integral $\int \frac{dx}{\sqrt{1-4x^2}}$.

Solution. Let $u = 2x$ and $du = 2\,dx$. Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-4x^2}} &= \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}} \\
&= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{2}\arcsin u + C \\
&= \frac{1}{2}\arcsin 2x + C.
\end{aligned}$$

Problem 3

Problem. Find the indefinite integral $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$.

Solution. Let $u = 2x$ and $du = 2\ dx$. Then

$$\begin{aligned}\int \frac{1}{x\sqrt{4x^2-1}}\ dx &= \int \frac{2}{2x\sqrt{(2x)^2-1}}\ dx \\ &= \int \frac{1}{u\sqrt{u^2-1}}\ du \\ &= \operatorname{arcsec} u + C \\ &= \operatorname{arcsec} 2x + C.\end{aligned}$$

Problem 4

Problem. Find the indefinite integral $\displaystyle\int \frac{12}{1+9x^2}\ dx$.

Solution. Let $u = 3x$ and $du = 3\ dx$. Then

$$\begin{aligned}\int \frac{12}{1+9x^2}\ dx &= 4\int \frac{3}{1+(3x)^2}\ dx \\ &= 4\int \frac{1}{1+u^2}\ du \\ &= 4\arctan u + C \\ &= 4\arctan 3x + C.\end{aligned}$$

Problem 5

Problem. Find the indefinite integral $\displaystyle\int \frac{1}{\sqrt{1-(x+1)^2}}\ dx$.

Solution. Let $u = x+1$ and $du = dx$. Then

$$\begin{aligned}\int \frac{1}{\sqrt{1-(x+1)^2}}\ dx &= \int \frac{1}{\sqrt{1-u^2}}\ du \\ &= \arcsin u + C \\ &= \arcsin{(x+1)} + C.\end{aligned}$$

Problem 6

Problem. Find the indefinite integral $\displaystyle\int \frac{1}{4+(x-3)^2}\ dx$.

Solution. You may want to do this in two stages. First, let $u = x-3$ and $du = dx$. Then

$$\int \frac{1}{4+(x-3)^2}\ dx = \int \frac{1}{4+u^2}\ du.$$

Next, let $u = 2v$ and $du = 2\ dv$. Then

$$\begin{aligned}
\int \frac{1}{4+u^2}\ du &= \int \frac{2}{4+(2v)^2}\ dv \\
&= \int \frac{2}{4+4v^2}\ dv \\
&= \frac{1}{2}\int \frac{1}{1+v^2}\ dv \\
&= \frac{1}{2}\arctan v + C \\
&= \frac{1}{2}\arctan \frac{u}{2} + C \\
&= \frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + C.
\end{aligned}$$

Of course, you may have anticipated all that and let $u = \frac{x-3}{2}$ from the start. That would work.

Problem 11

Problem. Find the indefinite integral $\displaystyle\int \frac{e^{2x}}{4+e^{4x}}\ dx$.

Solution. Recall that $e^{4x} = (e^{2x})^2$. Let $u = e^{2x}$ and $du = 2e^{2x}\ dx$. Then

$$\begin{aligned}
\int \frac{e^{2x}}{4+e^{4x}}\ dx &= \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\ dx \\
&= \frac{1}{2}\int \frac{1}{4+u^2}\ du
\end{aligned}$$

Now let $u = 2v$ and $du = 2\ dv$. Then

$$\begin{aligned}
\frac{1}{2}\int \frac{1}{4+u^2}\ du &= \frac{1}{2}\int \frac{2}{4+4v^2}\ dv \\
&= \frac{1}{4}\int \frac{1}{1+v^2}\ dv \\
&= \frac{1}{4}\arctan v + C \\
&= \frac{1}{4}\arctan \frac{u}{2} + C \\
&= \frac{1}{4}\arctan \frac{e^{2x}}{2} + C.
\end{aligned}$$

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx$$
$$= \frac{1}{2}\int \frac{1}{4+u^2}\,du$$

Problem 15

Problem. Find the indefinite integral $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$.

Solution. Let $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\,du$ (which will be more convenient, you just wait and see). Then

$$\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx = \int \frac{2u}{\sqrt{u^2}\sqrt{1-u^2}}\,du$$
$$= 2\int \frac{u}{u\sqrt{1-u^2}}\,du$$
$$= 2\int \frac{1}{\sqrt{1-u^2}}\,du$$
$$= 2\arcsin u + C$$
$$= 2\arcsin\sqrt{x} + C.$$

Problem 25

Problem. Evaluate the definite integral $\int_3^6 \frac{1}{25+(x-3)^2}\,dx$.

Solution. Let $u = x-3$ and $du = dx$. Note that $u(3) = 0$ and $u(6) = 3$. Then

$$\int_3^6 \frac{1}{25+(x-3)^2}\,dx = \int_0^3 \frac{1}{25+u^2}\,du.$$

Now let $u = 5v$ and $du = 5\ dv$. Note that $v(0) = 0$ and $v(3) = \frac{3}{5}$. Then

$$\begin{aligned}
\int_0^3 \frac{1}{25+u^2}\ du &= \int_0^{3/5} \frac{5}{25+25v^2}\ dv \\
&= \int_0^{3/5} \frac{1}{5+5v^2}\ dv \\
&= \frac{1}{5}\int_0^{3/5} \frac{1}{1+v^2}\ dv \\
&= \frac{1}{5}\left[\arctan v\right]_0^{3/5} \\
&= \frac{1}{5}\left(\arctan\frac{3}{5} - \arctan 0\right) \\
&= \frac{1}{5}\arctan\frac{3}{5}.
\end{aligned}$$

Problem 29

Problem. Evaluate the definite integral $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\ dx$.

Solution. Let $u = \cos x$ and $du = -\sin x\ dx$. Note that $u\left(\frac{\pi}{2}\right) = 0$ and $u(\pi) = -1$. Then

$$\begin{aligned}
\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\ dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\ dx \\
&= -\int_0^{-1} \frac{1}{1+u^2}\ du \\
&= -\left[\arctan u\right]_0^{-1} \\
&= -(\arctan(-1) - \arctan 0) \\
&= \frac{\pi}{4}.
\end{aligned}$$

Problem 33

Problem. Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{x^2-2x+2}$ by completing the square.

Solution. Complete the square.

$$\begin{aligned}
x^2 - 2x + 2 &= (x^2 - 2x + 1) + 1 \\
&= (x-1)^2 + 1.
\end{aligned}$$

Now the integral becomes $\displaystyle\int_0^2 \frac{dx}{1+(x-1)^2}$ and it is clear that we should make the substitution $u = x - 1$ and $du = dx$. We get

$$\begin{aligned}
\int_0^2 \frac{dx}{1+(x-1)^2} & \int_{-1}^1 \frac{du}{1+u^2} \\
&= [\arctan u]_{-1}^1 \\
&= \arctan 1 - \arctan(-1) \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Problem 37

Problem. Find the integral $\displaystyle\int \frac{1}{\sqrt{-x^2-4x}}\,dx$ by completing the square.

Solution. Complete the square.

$$\begin{aligned}
-x^2 - 4x &= -(x^2 + 4x) \\
&= -(x^2 + 4x + 4) + 4 \\
&= 4 - (x+2)^2.
\end{aligned}$$

Now the integral becomes $\displaystyle\int \frac{1}{\sqrt{4-(x+2)^2}}\,dx$ and it is clear that we should make the substitution $u = x + 2$ and $du = dx$. We get

$$\int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du.$$

Now let $u = 2v$ and $du = 2\,dv$. Then

$$\begin{aligned}
\int \frac{1}{\sqrt{4-u^2}}\,du &= \int \frac{2}{\sqrt{4-4v^2}}\,dv \\
&= \int \frac{1}{\sqrt{1-v^2}}\,dv \\
&= \arcsin v + C \\
&= \arcsin \frac{u}{2} + C \\
&= \arcsin\left(\frac{x+2}{2}\right) + C.
\end{aligned}$$